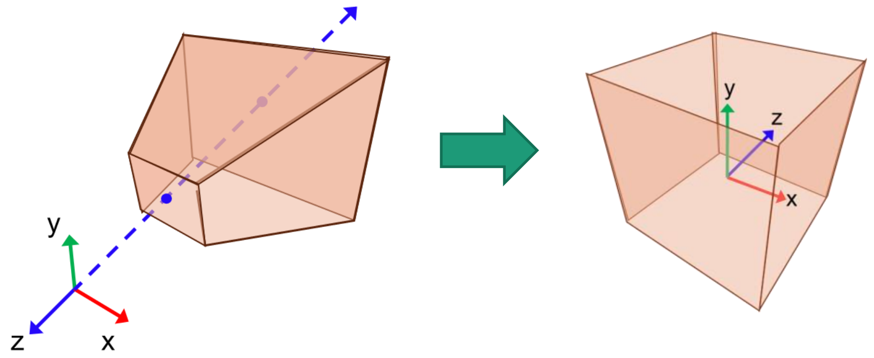
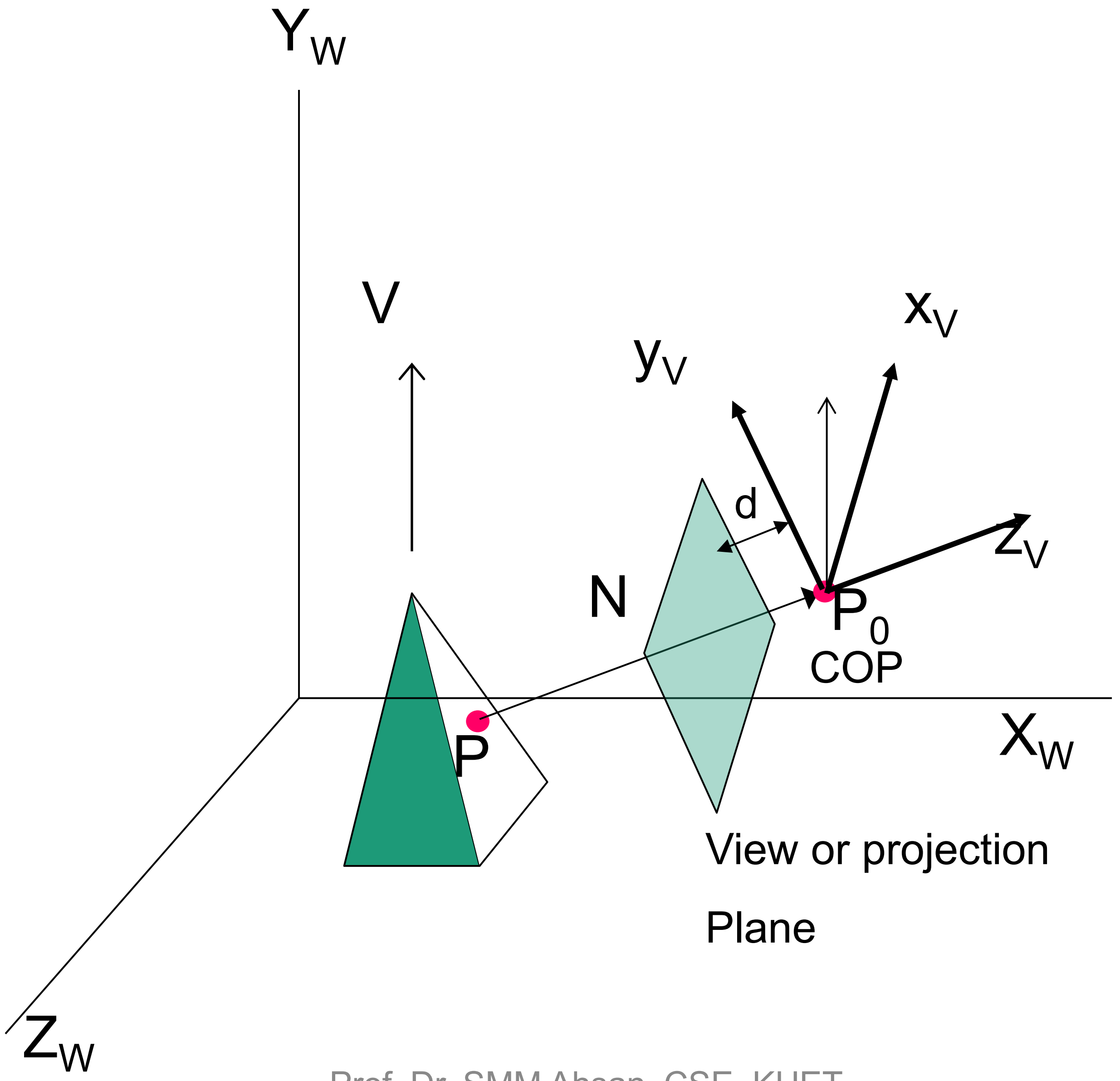


Projective Transformation

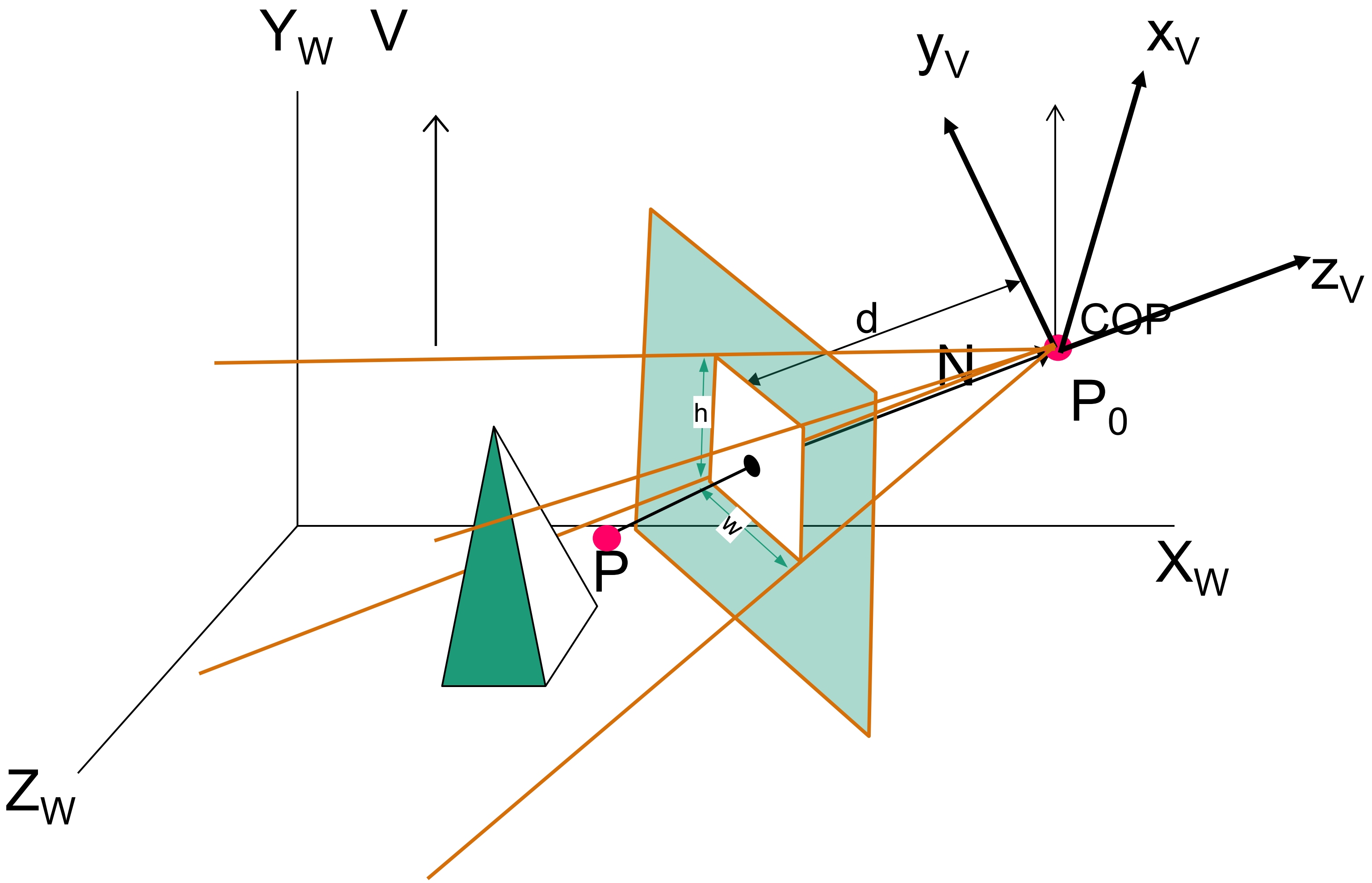
Amartya Kundu Durjoy

Lecturer, CSE, UGV

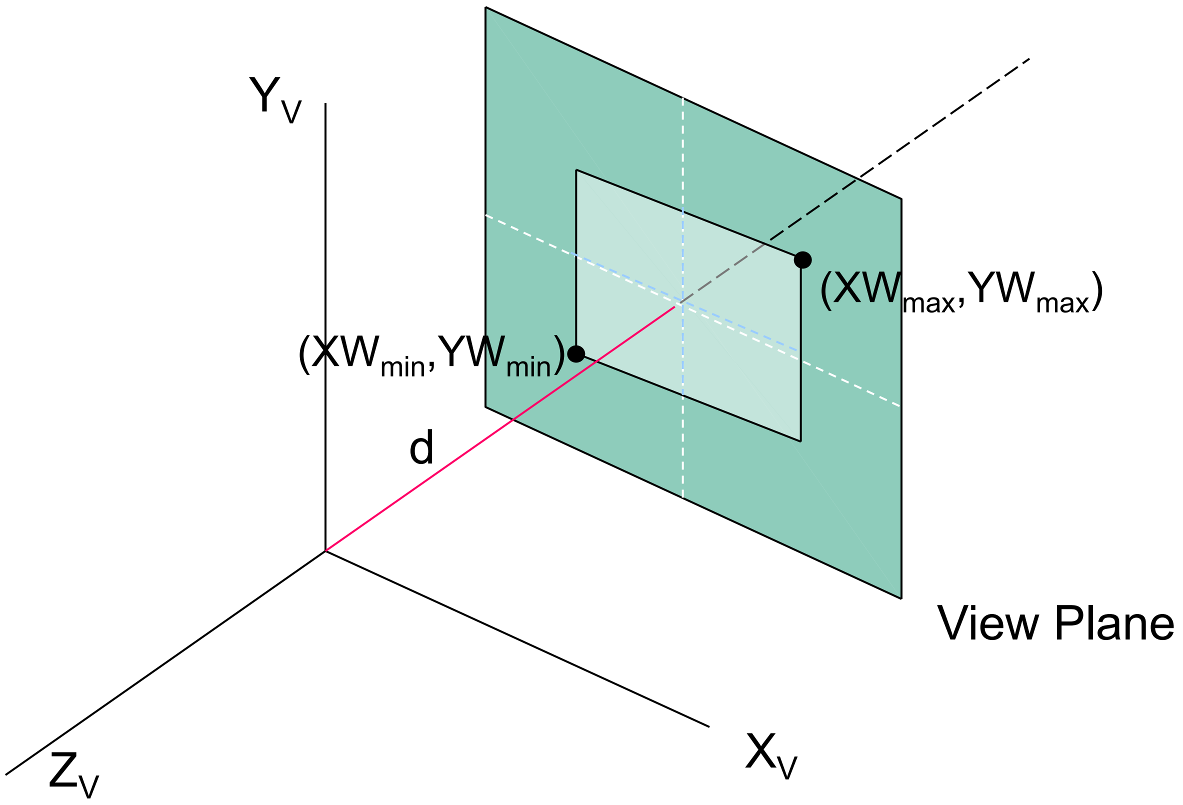
WCS, VCS and projection



View Window



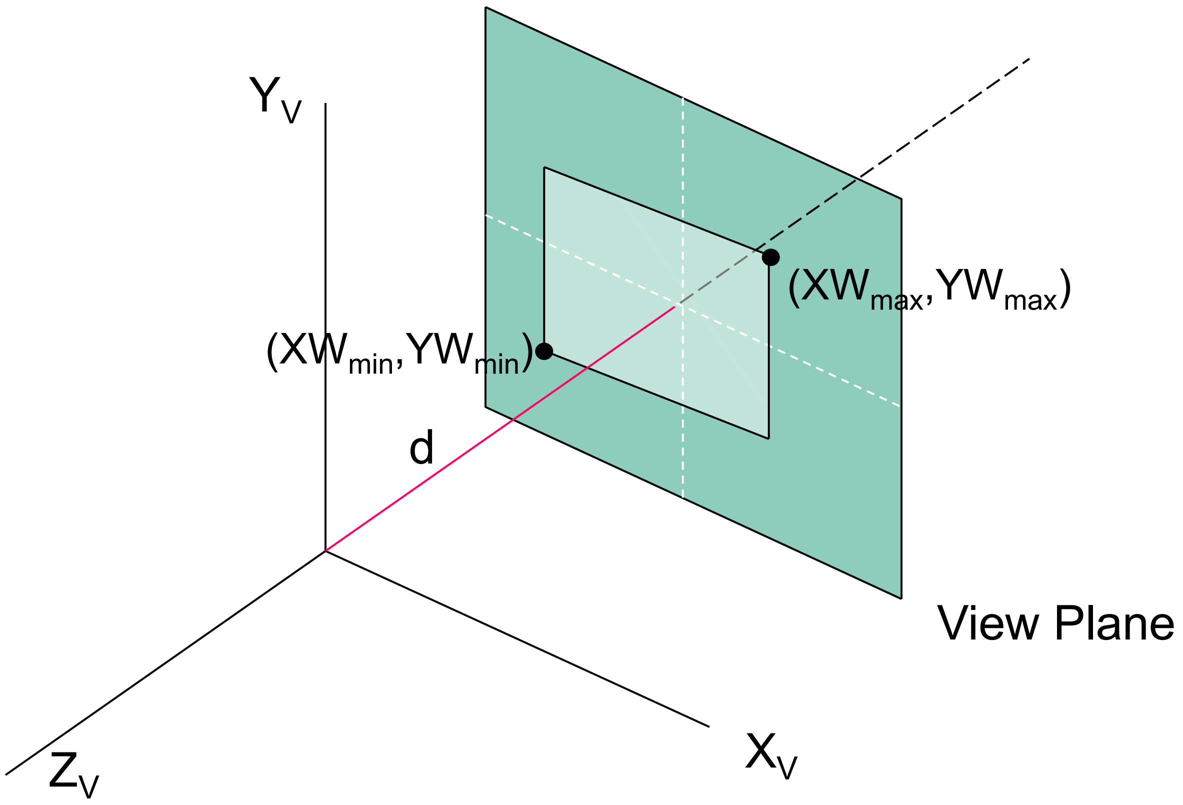
View Window - symmetric



$XW_{max} = –XW_{min}$,

$YW_{max} = –YW_{min}$

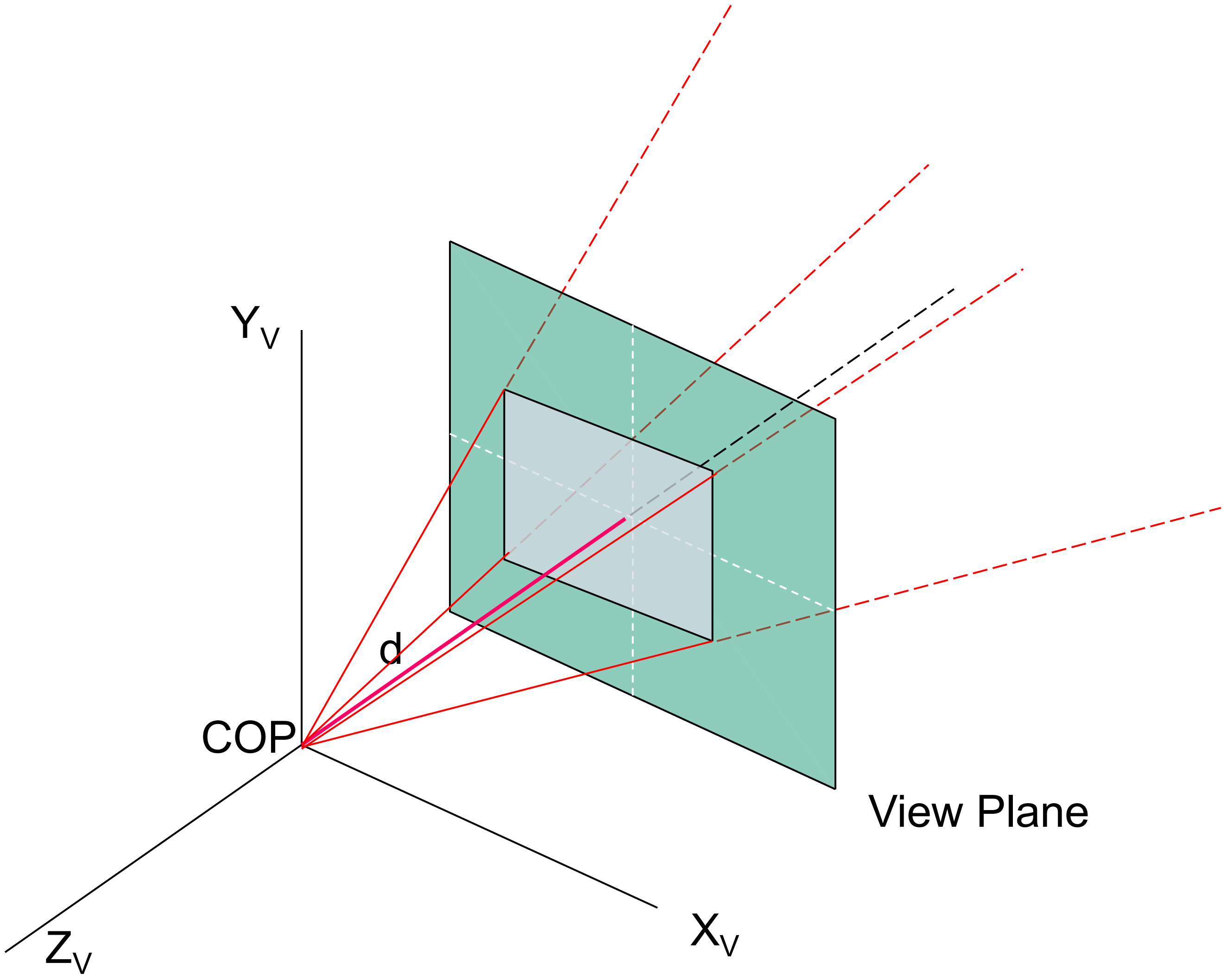
View Window - asymmetric



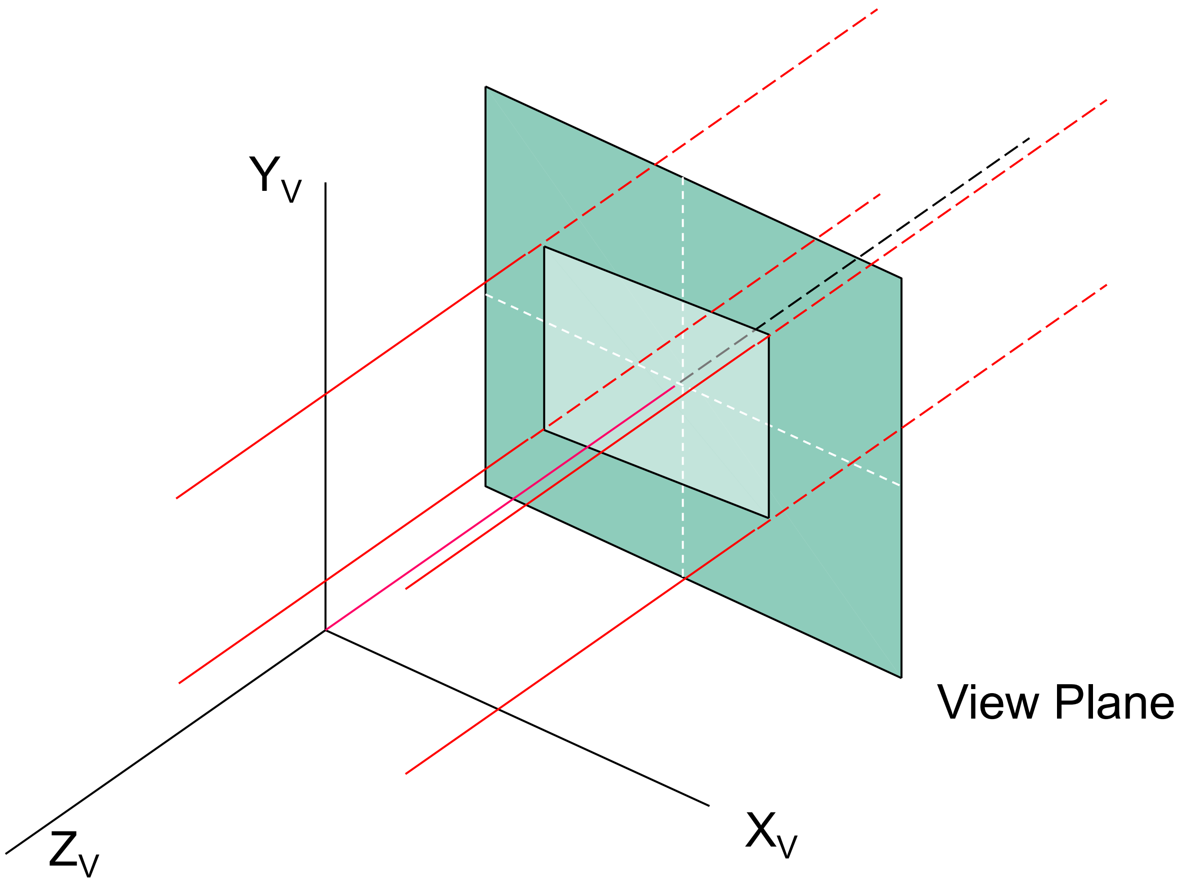
XW_{max} != $-XW_{min}$,

YW_{max} != $-YW_{min}$

View Volume - Perspective



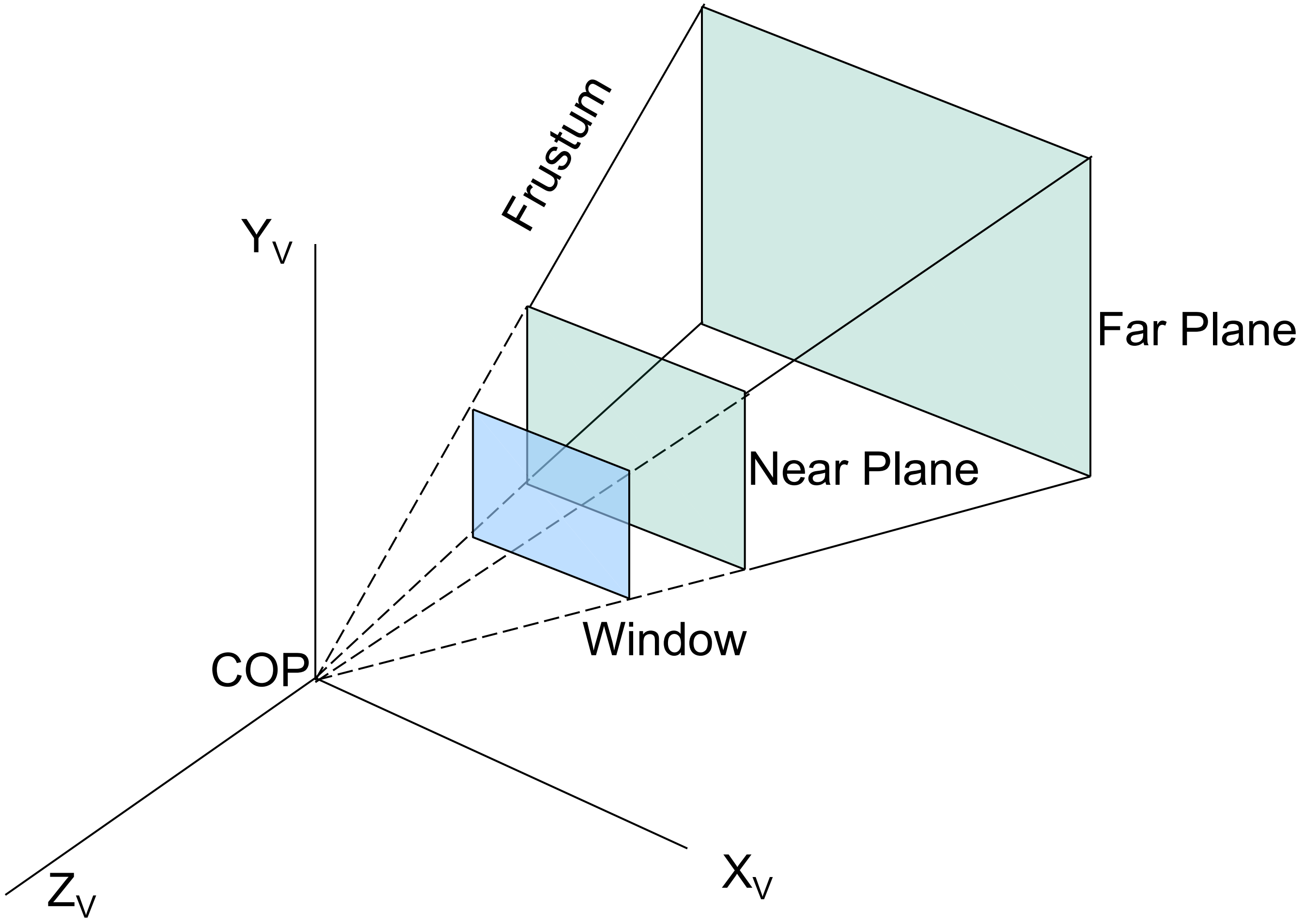
View Volume – Orthographic Projection



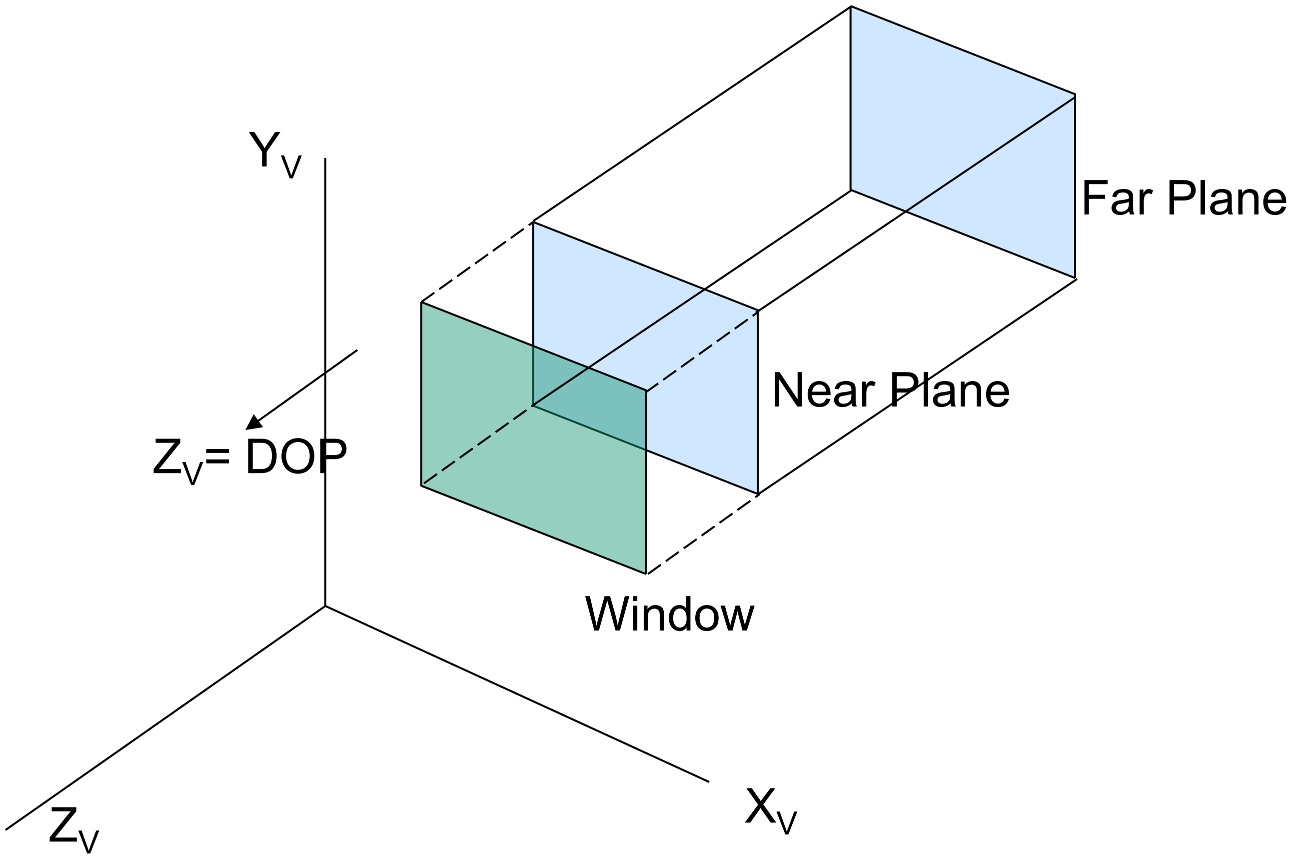
View Volume

- Finite ~ is obtained by limiting the extent of volume in Z_V direction
- Two plane parallel to view plane are used for this purpose, planes are –
  - **Near / Front plane** and **Far / Back plane**
  - Both must be same side of COP
- Adv. of using **Near, Far plane**
  - These planes eliminates parts of the scene from viewing operation based on depth.
  - Special adv. in case of perspective projection

Finite View Volume - Perspective

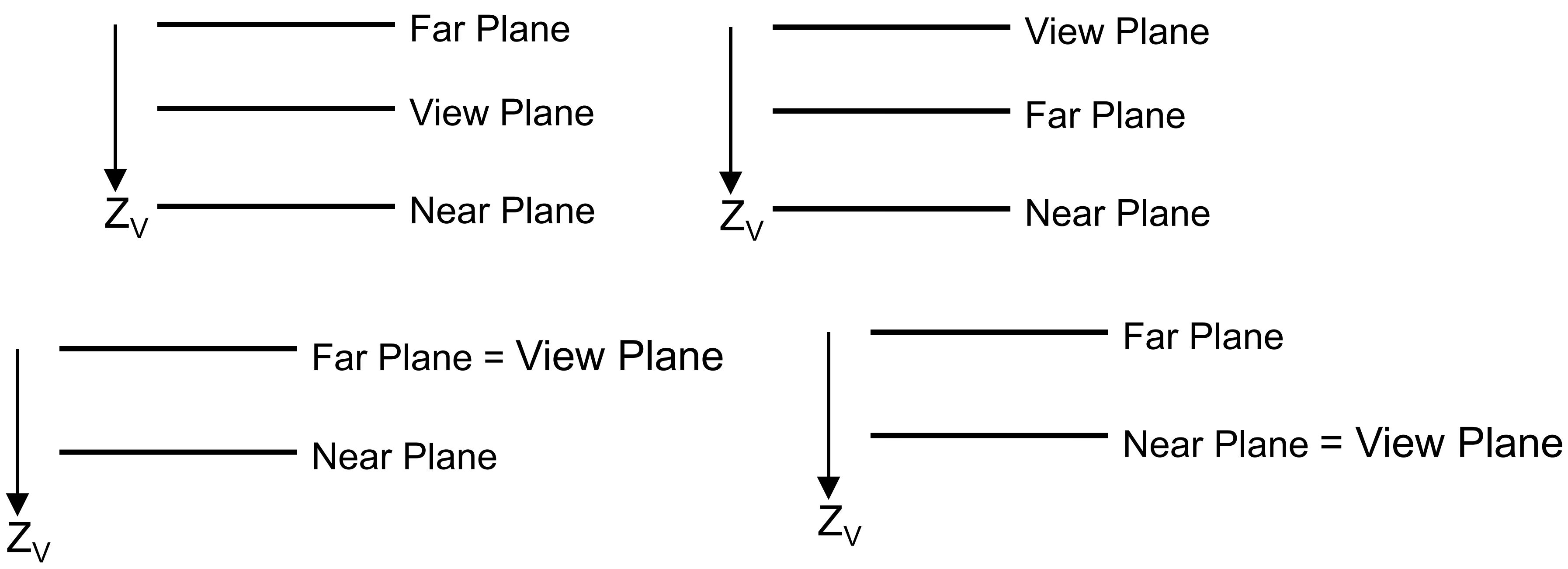


Finite View Volume – Orthographic Projection



Relative placement

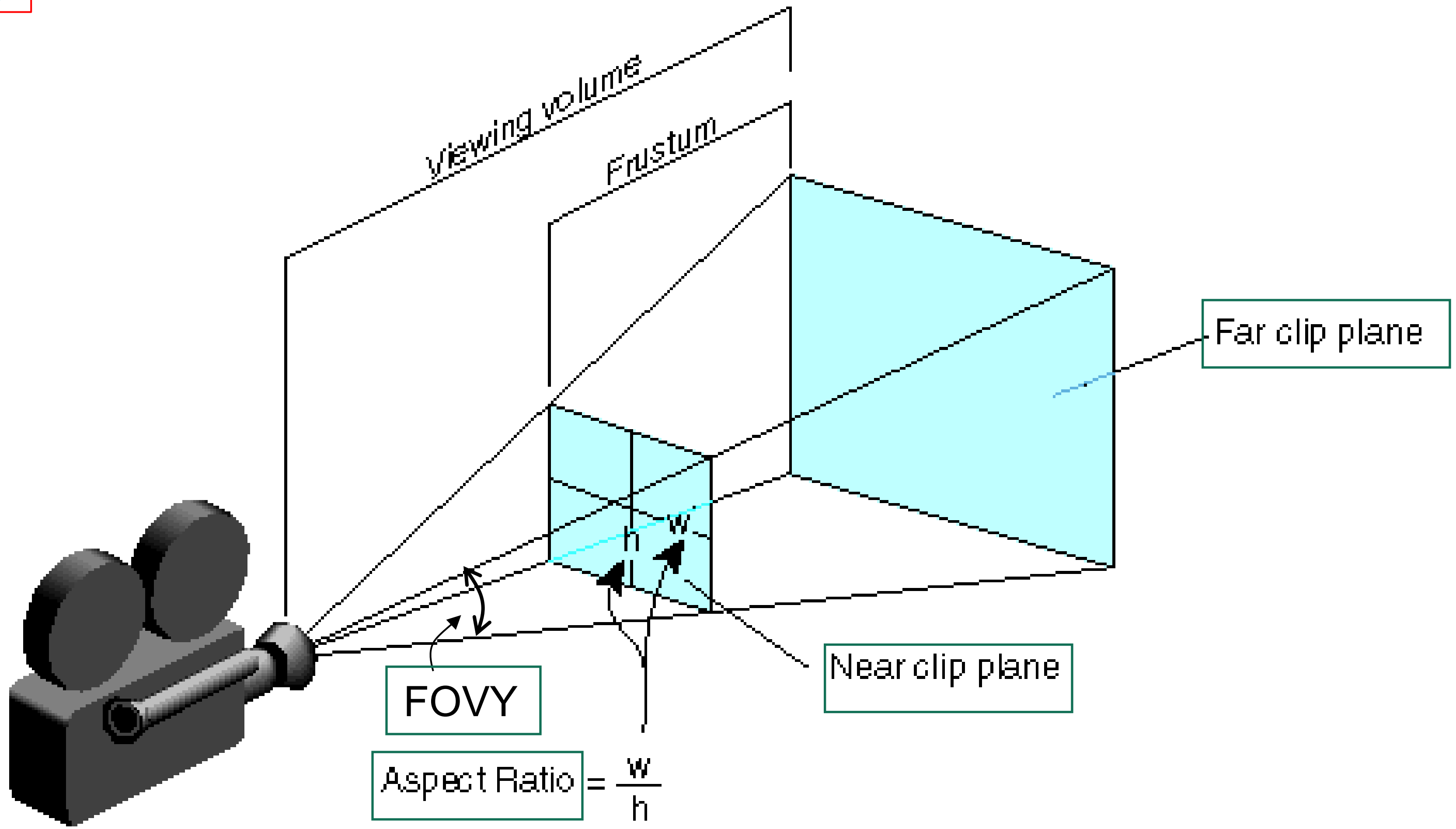
- ~ of **View, Near, Far planes** depend on
  - type of view we want &
  - limitation of Graphics package



OpenGL Uses it

Symmetric view volume - Perspective

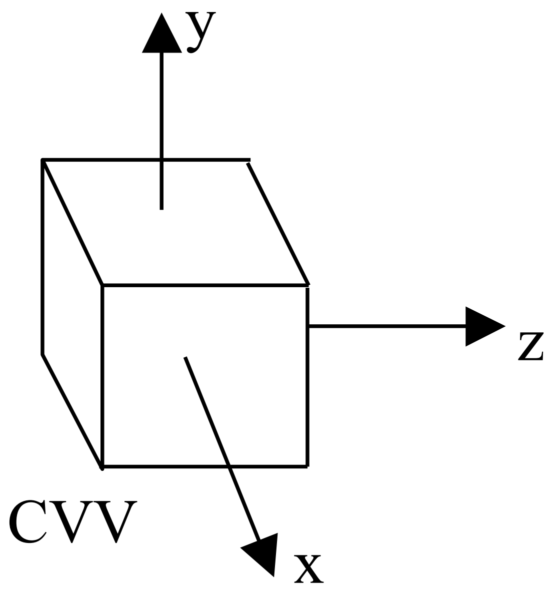
Another way



h= 2d * tan (fovy/2)

Normalization Transformation

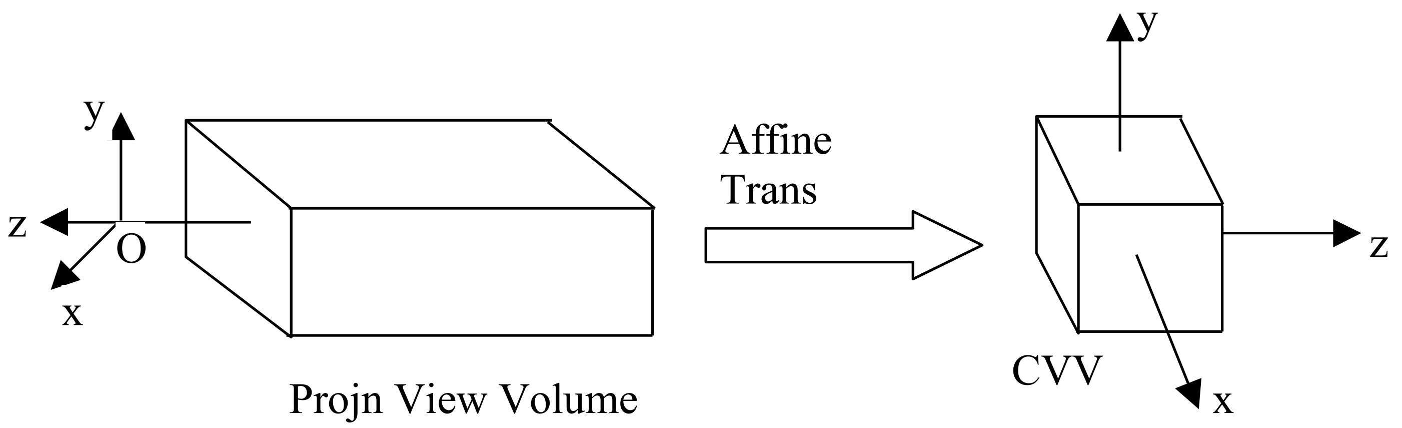
- Transforms VCS values to NDCS one
- Canonical View Volumes
  - The 3D object model is <u>not</u> actually clipped inside the projection view volume.
  - The view volume is instead mapped to a **canonical view volume** (CVV) which is a cube that extends from –1 to +1 in each dimension, having center at the origin.
  - The dimensions of the CVV facilitates a fast and efficient clipping.



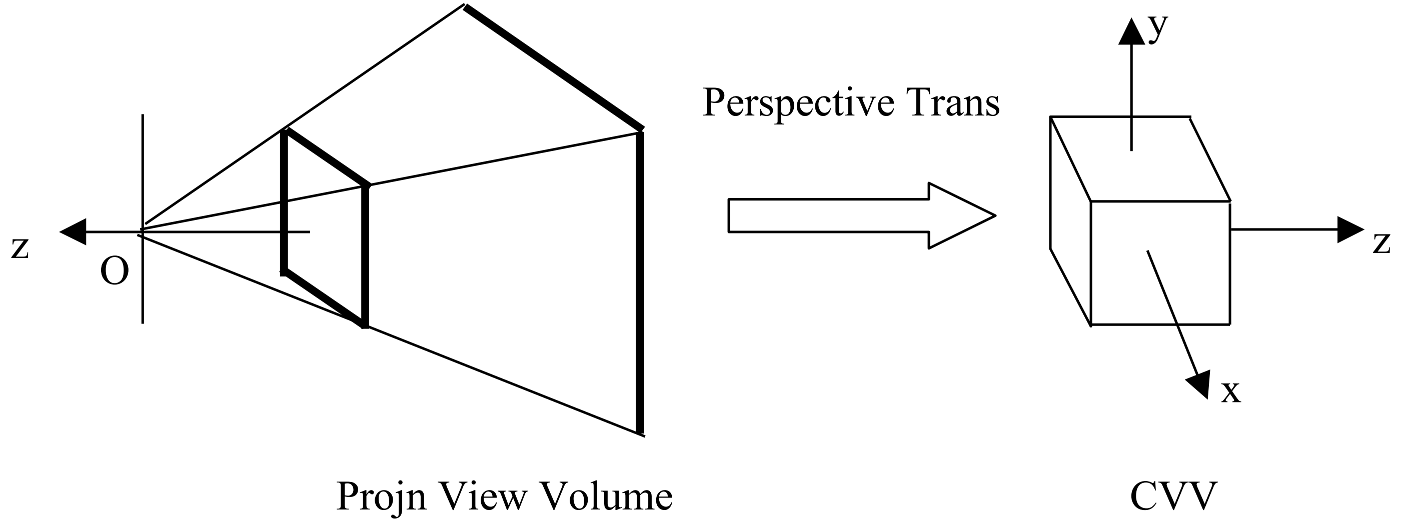
$$-1 \le x \le 1$$
$$-1 \le y \le 1$$
$$-1 \le z \le 1$$

Mapping to CVV

Orthographic:



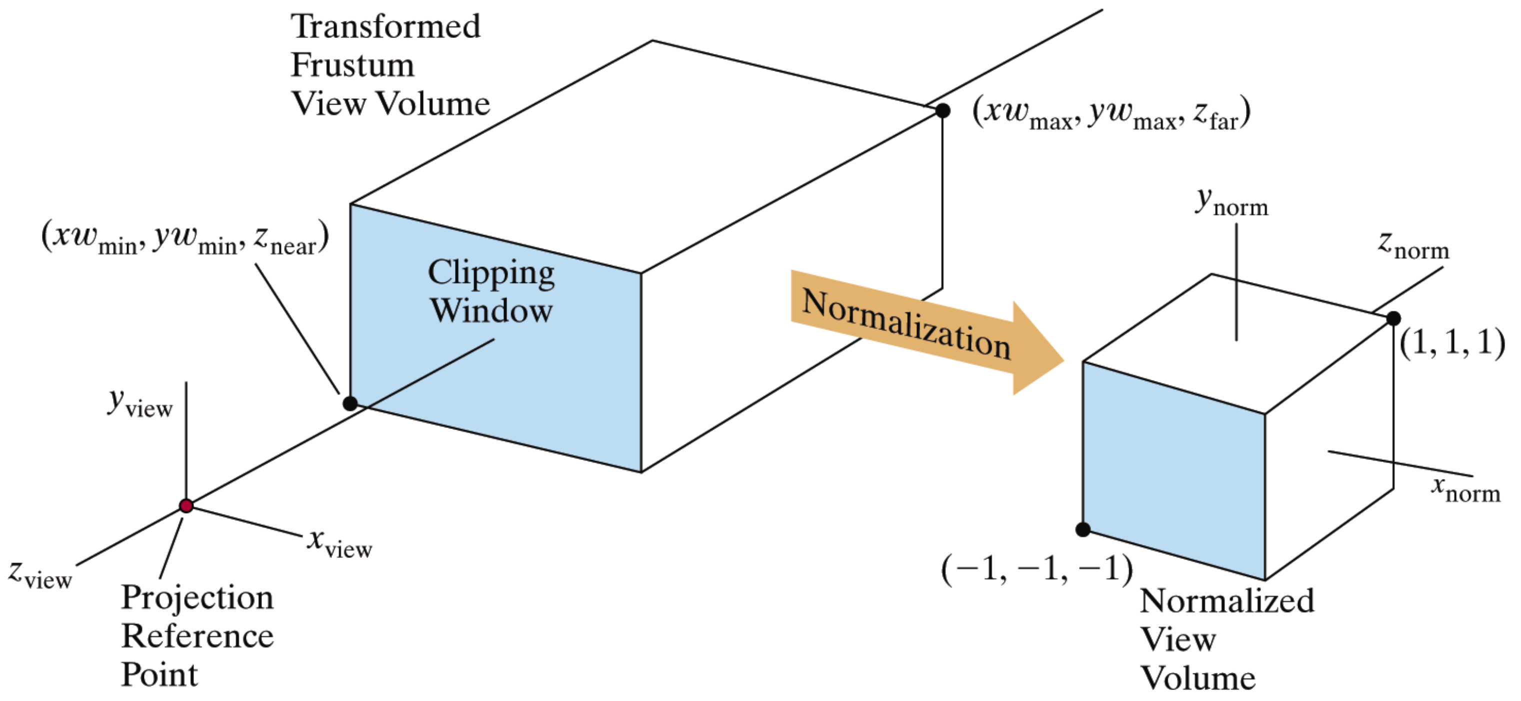
Perspective:



Normalization lets us clip against simple cube regardless of type of projection

Orthographic Transform

- The parallelepiped view volume is squeezed (scaled) around position (0, 0, 0) and the z axis is reversed

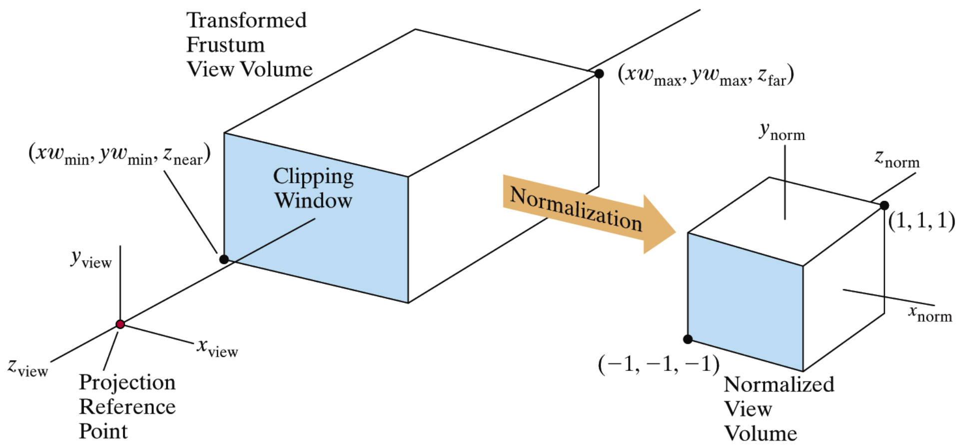


Orthographic Transform

$$T\left(-\frac{r+l}{2}, -\frac{t+b}{2}, \frac{|n|+|f|}{2}\right)$$

$$S\left(\frac{2}{r-l}, \frac{2}{t-b}, \frac{-2}{|f|-|n|}\right)$$

$$M_{orth,norm} = S.T$$



$$M_{orth,norm} = \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{|n|-|f|} & \frac{|n|+|f|}{|n|-|f|} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here,

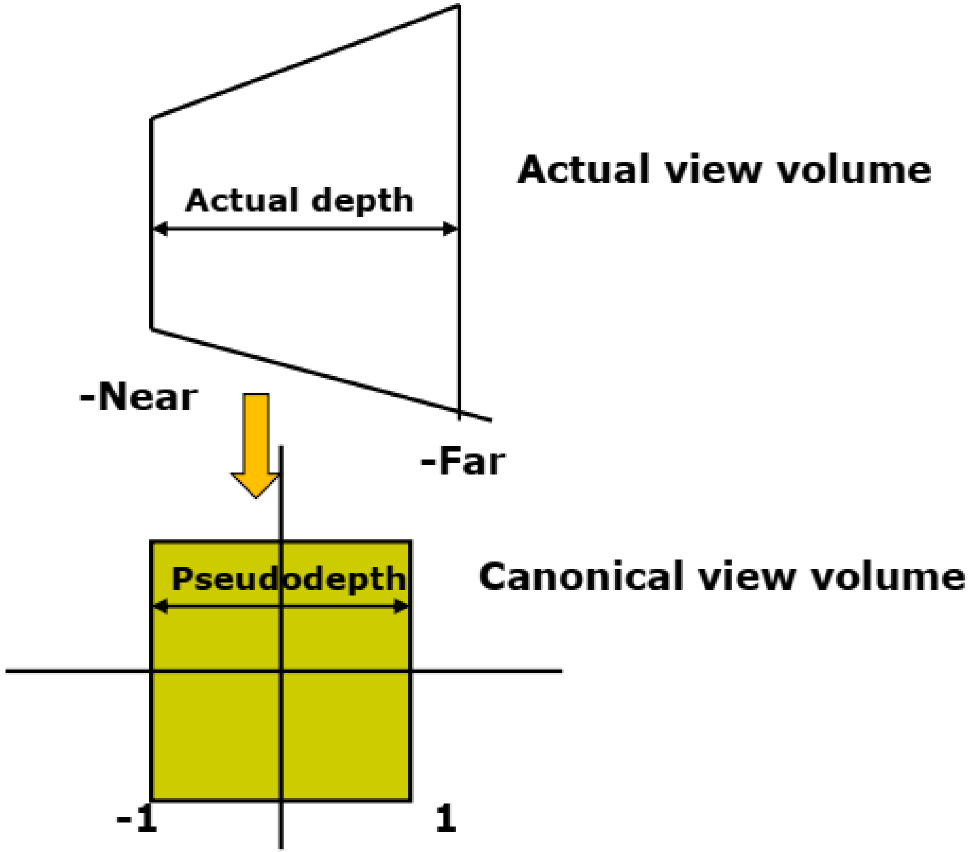
$f<n<0$

Or

$0<|n|<|f|$

Perspective Transform

- **Perspective transformation** maps actual z distance of perspective view volume to range [–1 to 1] (**Pseudodepth**) for canonical view volume

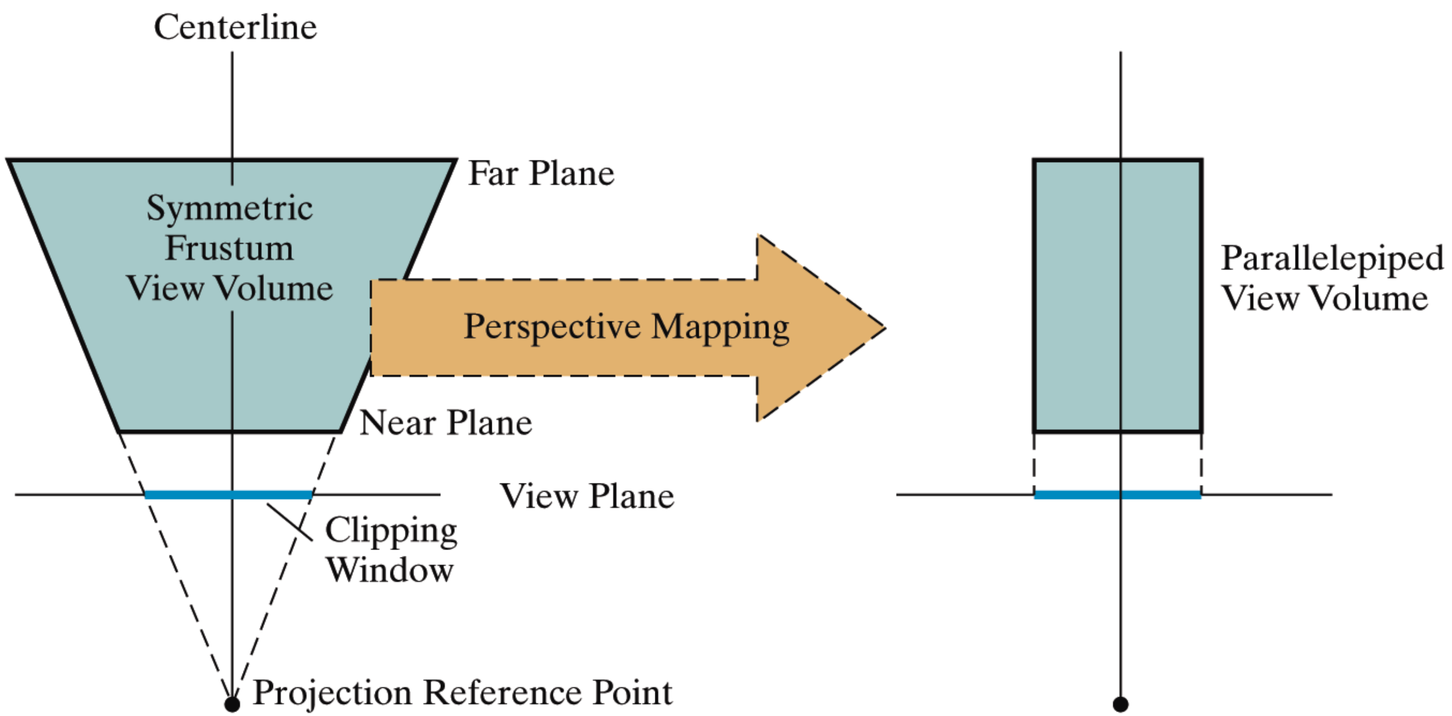


We want perspective Transformation and NOT classical projection!!

Set scaling z
Pseudodepth = az + b
Next solve for a and b

Perspective Transform

- The frustum shaped viewing volume has been converted to a parallelepiped
  - remember we preserved all z coordinate depth information
  - How?
- And then parallelepiped volume is mapped to CVV as before



Perspective without Depth

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{d} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ -\frac{z}{d} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ -\frac{z}{d} \end{pmatrix} \xrightarrow[\text{division}]{\text{perspective}} \begin{pmatrix} \frac{x}{-(z/d)} \\ \frac{y}{-(z/d)} \\ -d \\ 1 \end{pmatrix}$$

- The depth information is lost as the last two components are same
- But depth information of the projected points is essential for hidden surface removal and other purposes like blending, shading etc.

Perspective with Depth

- Sufficient to use **pseudodepth**
  - what is a good choice? Can we use z as pseudodepth? (farther point has more negative z value)
  - z'= $\alpha z + \beta$ which is a linear function before perspective division

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\frac{1}{d} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ \alpha z + \beta \\ -\frac{z}{d} \end{pmatrix} \xrightarrow{\text{perspective division}} \begin{pmatrix} \frac{x}{-(z/d)} \\ \frac{y}{-(z/d)} \\ -\frac{d(\alpha z + \beta)}{z} \\ 1 \end{pmatrix}$$

$$z' = -d \cdot \alpha - \frac{d \cdot \beta}{z}$$

For $\beta < 0$, z' is a monotonically increasing function of d*epth*.

Perspective Transform - Frustum to Parallelepiped

Rearrange the perspective transformation matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\frac{1}{d} & 0 \end{pmatrix} \cong \begin{pmatrix} |n| & 0 & 0 & 0 \\ 0 & |n| & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Perspective Transform - Frustum to Parallelepiped

Chose α, β, such a way that a point in near plane / far plane perspectively projects on near / far plane

$$\begin{pmatrix} |n| & 0 & 0 & 0 \\ 0 & |n| & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} r & rf/n \\ t & tf/n \\ -|n| & -|f| \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} |n|r & rf \\ |n|t & tf \\ -\alpha|n|+\beta & -\alpha|f|+\beta \\ |n| & |f| \end{pmatrix} = \begin{pmatrix} r & r \\ t & t \\ \frac{-\alpha|n|+\beta}{|n|} & \frac{-\alpha|f|+\beta}{|f|} \\ 1 & 1 \end{pmatrix}$$

Apply Boundary Condition

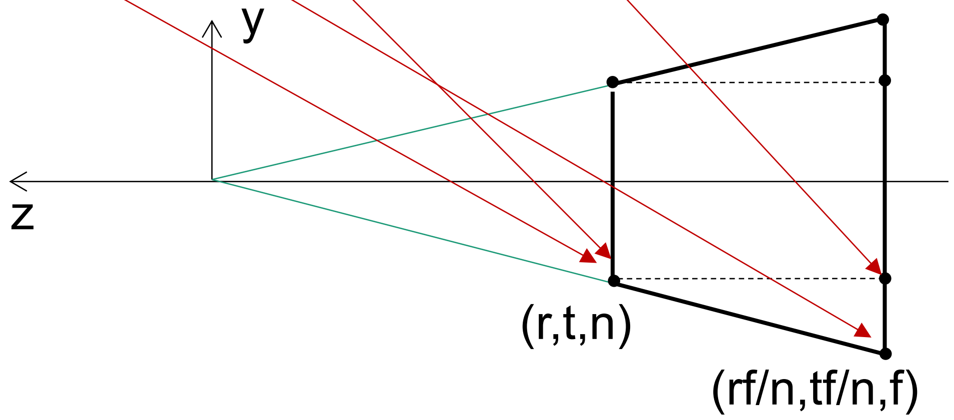
$$\frac{-\alpha|n|+\beta}{|n|} = -|n|$$
$$=> -\alpha|n|+\beta = -|n|^2$$

Similarly

$$-\alpha|f|+\beta = -|f|^2$$

Solving

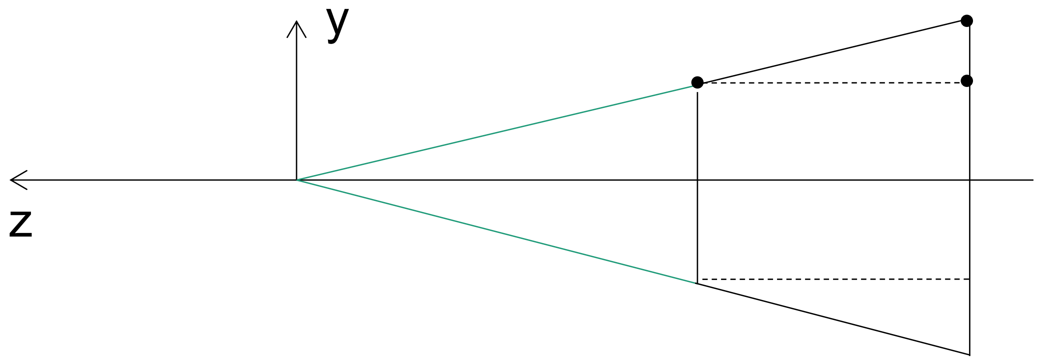
$$\alpha = |n|+|f|$$
$$\beta = |n||f|$$



Perspective Transform - Parallelepiped to CVV

$$M_{per,norm} = \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{|n|-|f|} & \frac{|n|+|f|}{|n|-|f|} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |n| & 0 & 0 & 0 \\ 0 & |n| & 0 & 0 \\ 0 & 0 & |n|+|f| & |n||f| \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} \frac{2|n|}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2|n|}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{|n|+|f|}{|n|-|f|} & \frac{2|f||n|}{|n|-|f|} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$



Perspective transform

If window is symmetric, $r = -l$ and $t = -b$, then

$$M_{per,norm} = \begin{bmatrix} \frac{2|n|}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2|n|}{t-b} & 0 & 0 \\ 0 & 0 & \frac{|n|+|f|}{|n|-|f|} & \frac{2|f||n|}{|n|-|f|} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

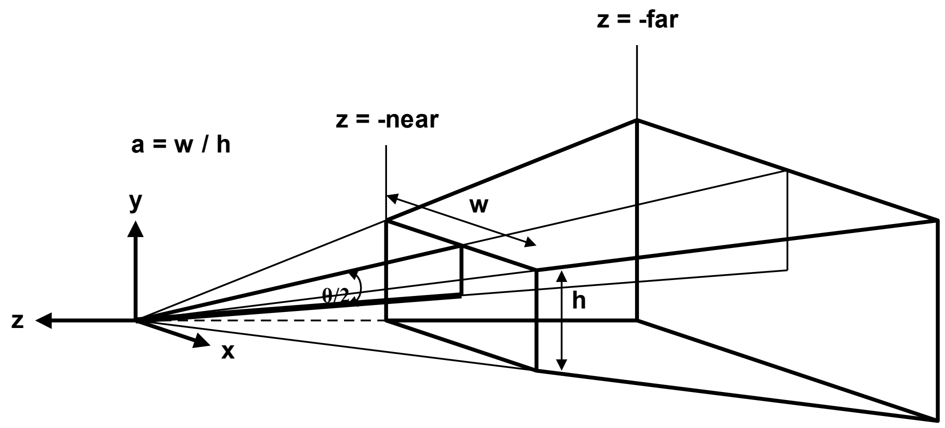
symmetric

$$M_{per,norm} = \begin{bmatrix} \frac{2|n|}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2|n|}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{|n|+|f|}{|n|-|f|} & \frac{2|f||n|}{|n|-|f|} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Asymmetric / general

Perspective transform

$$M_{per,norm} = \begin{bmatrix} \frac{2|n|}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2|n|}{t-b} & 0 & 0 \\ 0 & 0 & \frac{|n|+|f|}{|n|-|f|} & \frac{2|f||n|}{|n|-|f|} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$



Window is symmetric, but defined by aspect ratio a, fovy θ, then

$$c = \cot\left(\frac{\theta}{2}\right) = \frac{|n|}{t} \Rightarrow t = \frac{|n|}{c}; \quad b = -\frac{|n|}{c}$$

$$a = \frac{w}{h} = \frac{r}{t} \Rightarrow r = at = \frac{a|n|}{c}; \quad l = -\frac{a|n|}{c}$$

$$M_{per,norm} = \begin{bmatrix} \frac{c}{a} & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & \frac{|n|+|f|}{|n|-|f|} & \frac{2|f||n|}{|n|-|f|} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Perspective Transformation Matrix

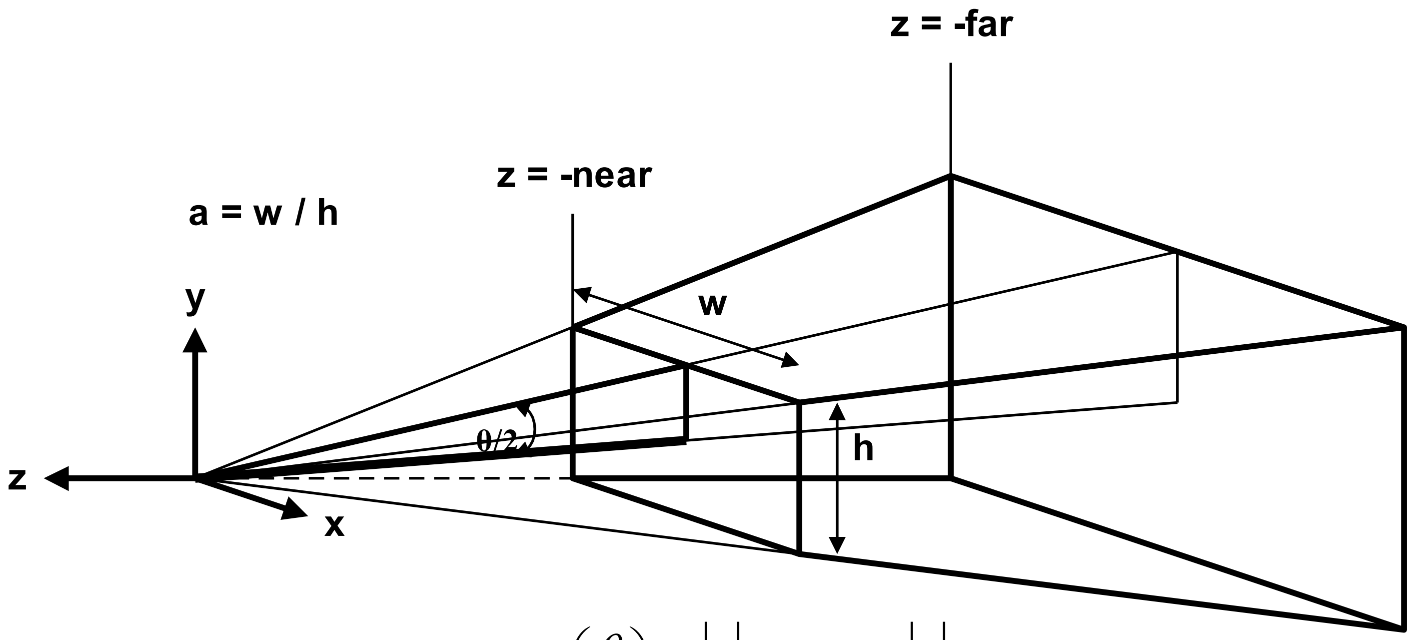
Another way

The matrix to perform ***perspective transformation***:

$$\begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma \cdot x \\ \delta \cdot y \\ \alpha \cdot z + \beta \\ -z \end{pmatrix}$$

$$= \begin{pmatrix} \gamma \cdot x/(-z) \\ \delta \cdot y/(-z) \\ (\alpha \cdot z + \beta)/(-z) \\ 1 \end{pmatrix}$$

Perspective Transformation Matrix - Symmetric

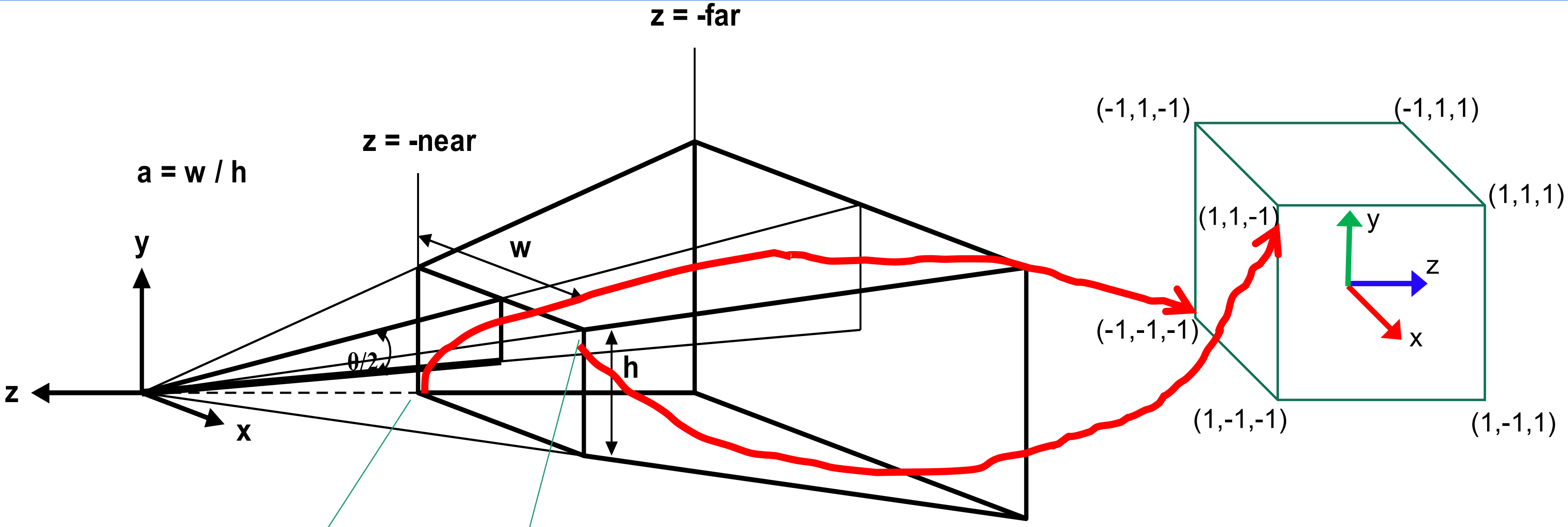


$$c = \cot\left(\frac{\theta}{2}\right) = \frac{|z|}{|y|} \Rightarrow |y| = \frac{|z|}{c}$$

$$a = \frac{w}{h} = \frac{|x|}{|y|} \Rightarrow |x| = a|y| = \frac{a|z|}{c}$$

$$\therefore (x, y, z) \Rightarrow \left(\pm\frac{a|z|}{c}, \pm\frac{|z|}{c}, z\right)$$

Perspective Transformation Matrix - Symmetric



$$\left(-\frac{a|n|}{c},-\frac{|n|}{c},-|n|\right)\rightarrow(-1,-1,-1)$$

$$\left(-\frac{a|n|}{c},\frac{|n|}{c},-|n|\right)\rightarrow(-1,1,-1)$$

$$\left(\frac{a|n|}{c},-\frac{|n|}{c},-|n|\right)\rightarrow(1,-1,-1)$$

$$\left(\frac{a|n|}{c},\frac{|n|}{c},-|n|\right)\rightarrow(1,1,-1)$$

$$\left(-\frac{af}{c},-\frac{f}{c},-f\right)\rightarrow(-1,-1,1)$$

$$\left(-\frac{af}{c},\frac{f}{c},-f\right)\rightarrow(-1,1,1)$$

$$\left(\frac{af}{c},-\frac{f}{c},-f\right)\rightarrow(1,-1,1)$$

$$\left(\frac{af}{c},\frac{f}{c},-f\right)\rightarrow(1,1,1)$$

all f are $|f|$

Perspective Transformation Matrix - Symmetric

$$\left(-\frac{an}{c},-\frac{n}{c},-n\right)\rightarrow(-1,-1,-1)$$

$$\begin{pmatrix}\gamma & 0 & 0 & 0\\ 0 & \delta & 0 & 0\\ 0 & 0 & \alpha & \beta\\ 0 & 0 & -1 & 0\end{pmatrix}\cdot\begin{pmatrix}-\frac{an}{c}\\ -\frac{n}{c}\\ -n\\ 1\end{pmatrix}=\begin{pmatrix}-1\\ -1\\ -1\\ 1\end{pmatrix} \Leftrightarrow \begin{pmatrix}-\frac{\gamma an}{c}\\ -\frac{\delta n}{c}\\ -\alpha n+\beta\\ n\end{pmatrix}=\begin{pmatrix}-1\\ -1\\ -1\\ 1\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}-\frac{\gamma a}{c}\\ -\frac{\delta}{c}\\ \frac{-\alpha n+\beta}{n}\\ 1\end{pmatrix}=\begin{pmatrix}-1\\ -1\\ -1\\ 1\end{pmatrix}$$

$$\Rightarrow \gamma=\frac{c}{a}$$

$$\Rightarrow \delta=c$$

$$\Rightarrow -\alpha n+\beta=-n$$

Perspective Transformation Matrix - Symmetric

$$\left(-\frac{af}{c},-\frac{f}{c},-f\right)\rightarrow(-1,-1,1)$$

$$\begin{pmatrix}\gamma & 0 & 0 & 0\\ 0 & \delta & 0 & 0\\ 0 & 0 & \alpha & \beta\\ 0 & 0 & -1 & 0\end{pmatrix}\cdot\begin{pmatrix}-\frac{af}{c}\\ -\frac{f}{c}\\ -f\\ 1\end{pmatrix}=\begin{pmatrix}-1\\ -1\\ 1\\ 1\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}-\frac{\gamma af}{c}\\ -\frac{\delta f}{c}\\ -\alpha f+\beta\\ f\end{pmatrix}=\begin{pmatrix}-1\\ -1\\ -1\\ 1\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}-\frac{\gamma a}{c}\\ -\frac{\delta}{c}\\ \frac{-\alpha f+\beta}{f}\\ 1\end{pmatrix}=\begin{pmatrix}-1\\ -1\\ 1\\ 1\end{pmatrix}$$

$$\Rightarrow -\alpha f+\beta=f$$

$$\because -\alpha n+\beta=-n$$

$$\therefore \alpha=\frac{f+n}{n-f}$$

$$\therefore \beta=\frac{2fn}{n-f}$$

Perspective Transformation Matrix - Symmetric

The matrix to perform ***perspective transformation***:

$$\begin{pmatrix} \frac{c}{a} & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & \frac{f+n}{n-f} & \frac{2fn}{n-f} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Perspective projection

- Using the Frustum parameters, and considering it a symmetric one
- Final matrix

$$\begin{bmatrix} \frac{2n}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2n}{t-b} & 0 & 0 \\ 0 & 0 & \frac{f+n}{n-f} & \frac{2fn}{n-f} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

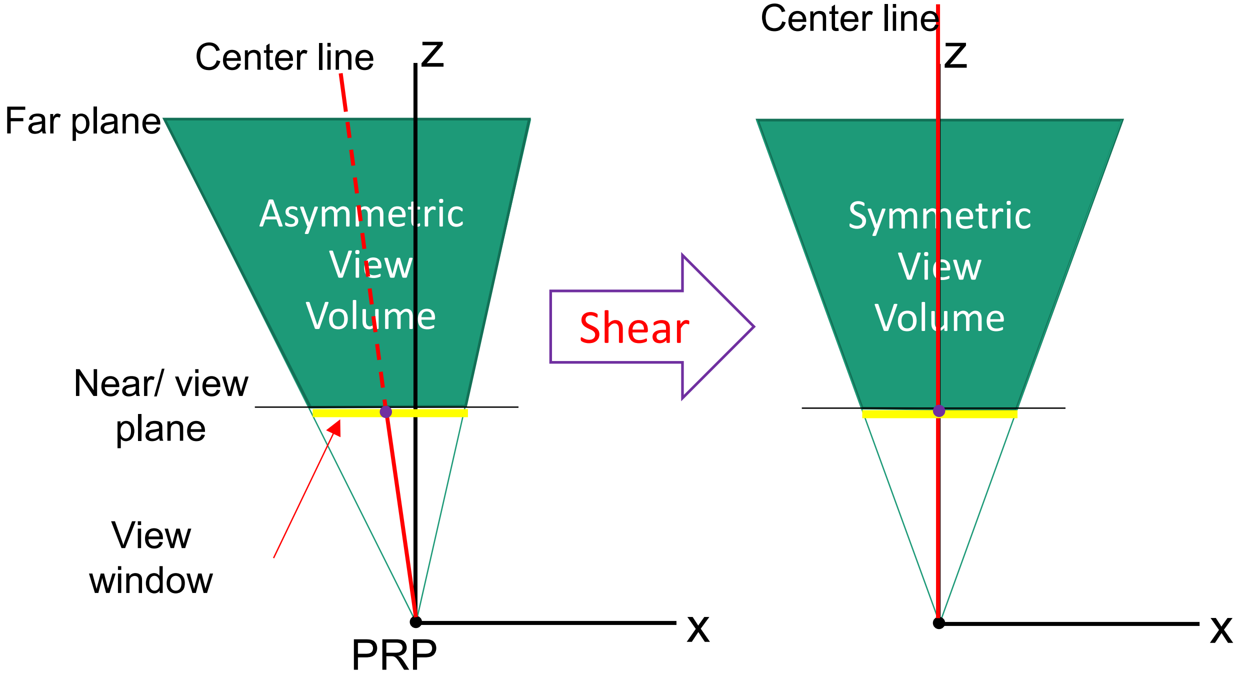
$$\begin{bmatrix} \frac{-2n}{r-l} & 0 & 0 & 0 \\ 0 & \frac{-2n}{t-b} & 0 & 0 \\ 0 & 0 & \frac{f+n}{n-f} & \frac{-2fn}{n-f} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Baker Book

This is also true, here all n and f are absolute values

Perspective projection

- The view frustum may not be centered along the view vector, so it needs one more step, first shear the window.



$$\begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Shear matrix

Shear

$x' = x + az$

$y' = y + bz$

$((r+l)/2, (t+b)/2, -n) \longrightarrow (0, 0, -n)$

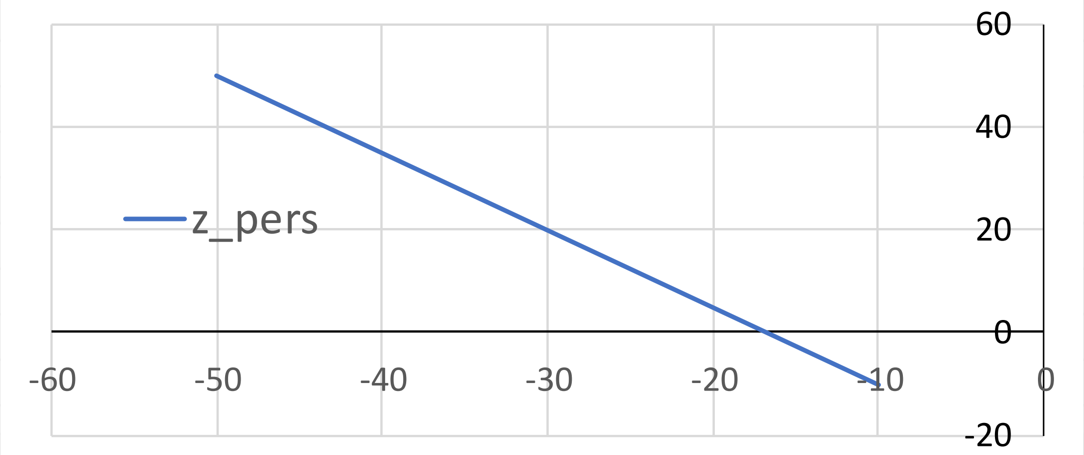
$a = (r+l)/2n$

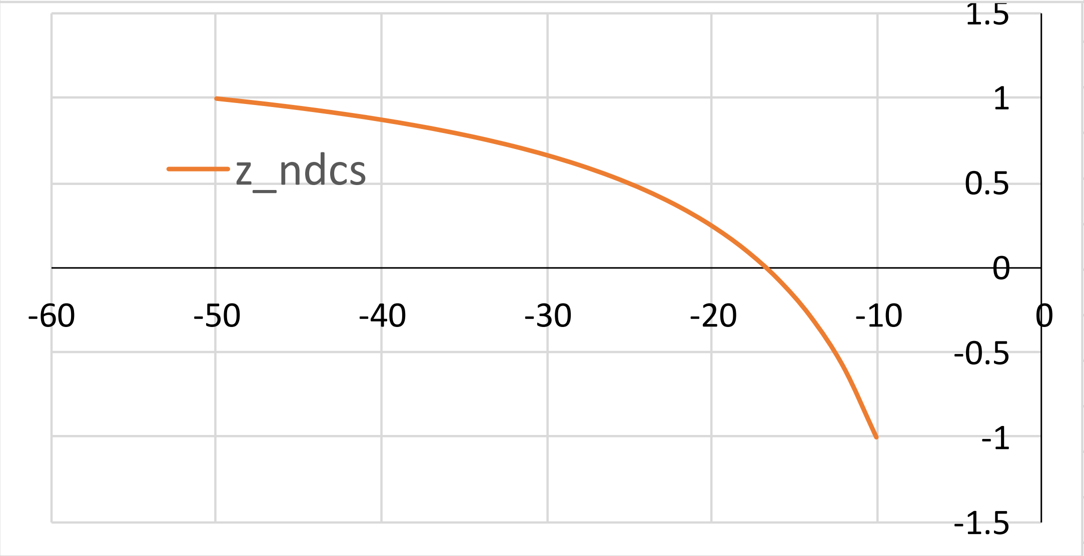
$b = (t+b)/2n$

Perspective projection

n	-10	alpha	-1.5
f	-50	beta	-25

z_eye	z_pers	z_ndcs	p_divisor
-10	-10	-1	10
-12	-7	-0.58333	12
-14	-4	-0.28571	14
-16	-1	-0.0625	16
-18	2	0.111111	18
-20	5	0.25	20
-22	8	0.363636	22
-24	11	0.458333	24
-26	14	0.538462	26
-28	17	0.607143	28
-30	20	0.666667	30
-32	23	0.71875	32
-34	26	0.764706	34
-36	29	0.805556	36
-38	32	0.842105	38
-40	35	0.875	40
-42	38	0.904762	42
-44	41	0.931818	44
-46	44	0.956522	46
-48	47	0.979167	48
-50	50	1	50

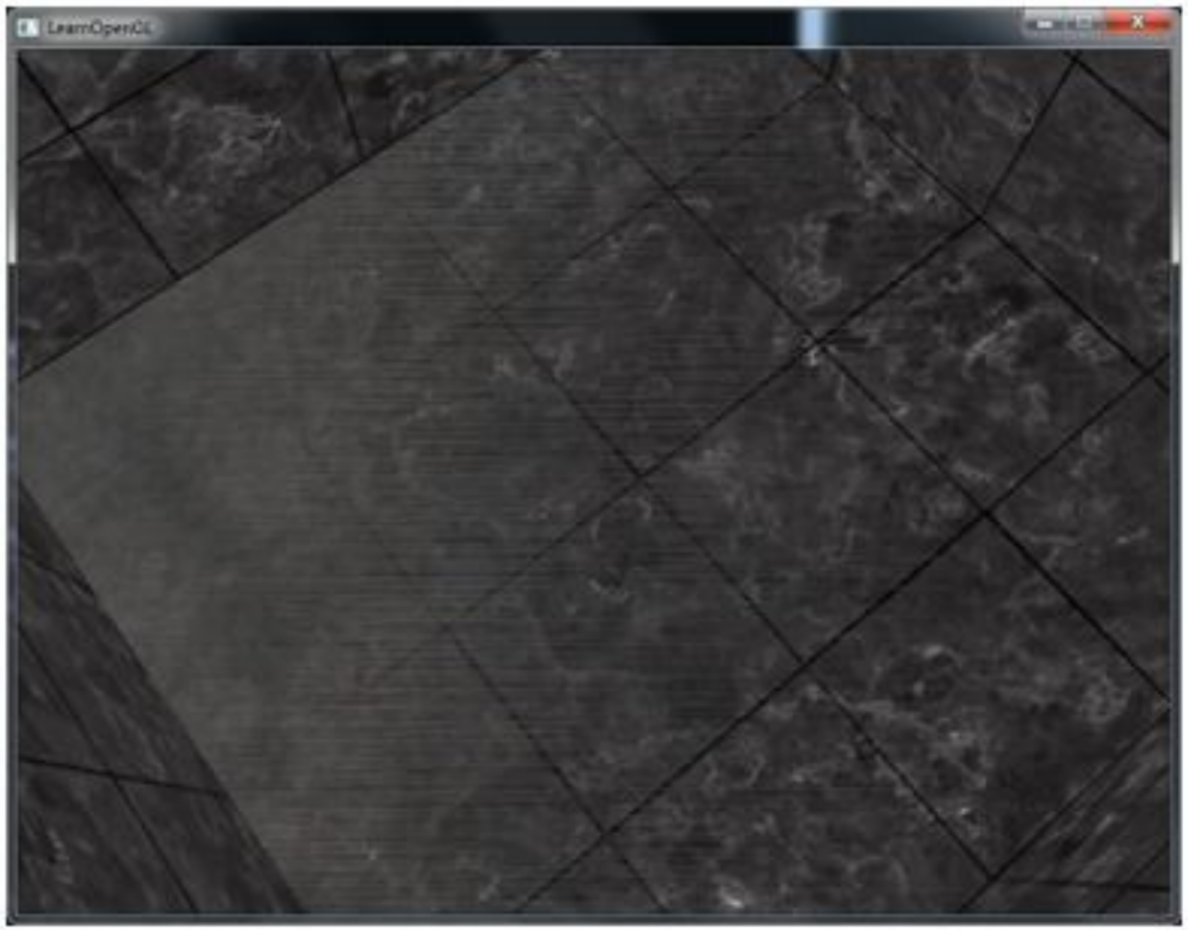




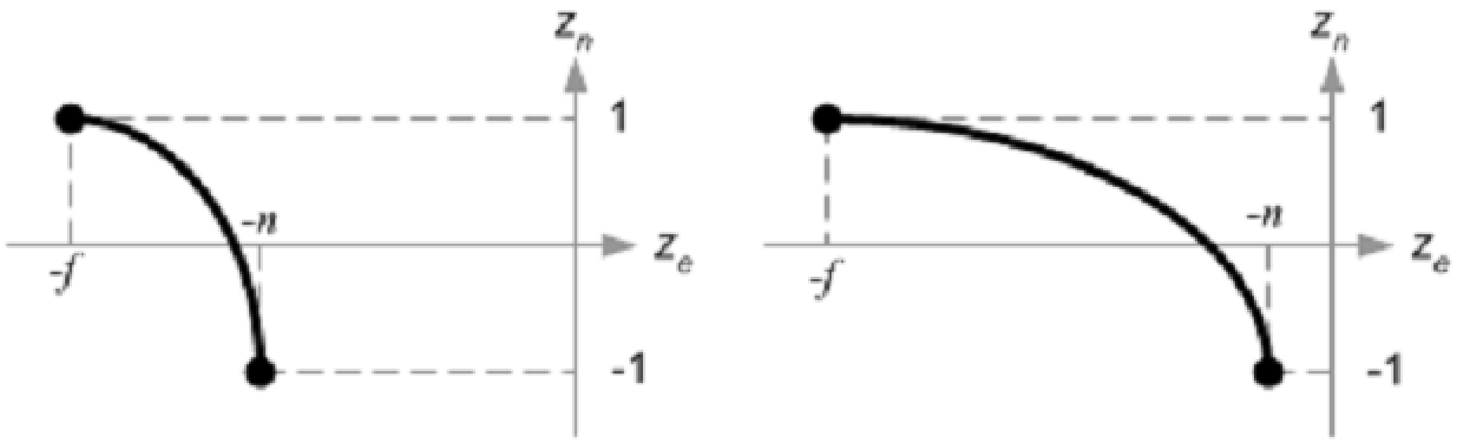
Z-Fighting Problem

- A common visual artifact might occur when two planes are closely aligned to each
  - The depth buffer does not have enough precision to figure out which one is in front of the other.

$$z' = -d \cdot \alpha - \frac{d \cdot \beta}{z}$$



Z-Fighting Problem



Comparison of Depth Buffer Precisions

$$z' = -d \cdot \alpha - \frac{d \cdot \beta}{z}$$

Z	Z′
97	0.9993752
98	0.9995877
99	0.9997959

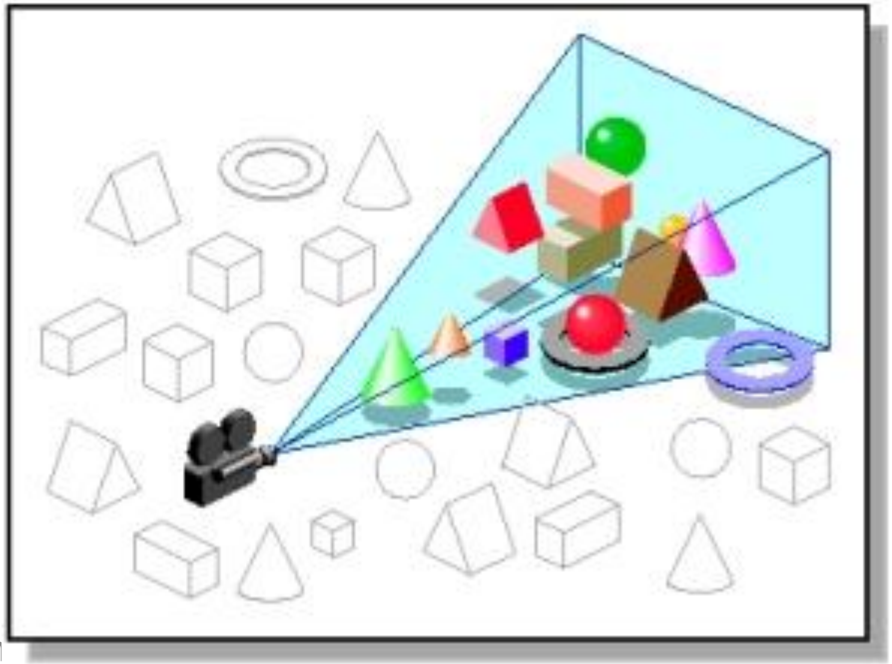
- N=1, F=100, z' ranges from -1 to +1
- Prevention
  - *set the near plane as far as possible*
  - *never place objects too close to each other in a way that some of them closely overlap*
  - *use a higher precision depth buffer*

WCS, VCS, NDCS and projection

- You are given info about
  - P_0, P, V in WCS
  - d, f, a=h/w, fovy
- Let the apex of the pyramidal object in the scene $(1,2,1)_{WCS}$
- Find the NDCS value of the apex

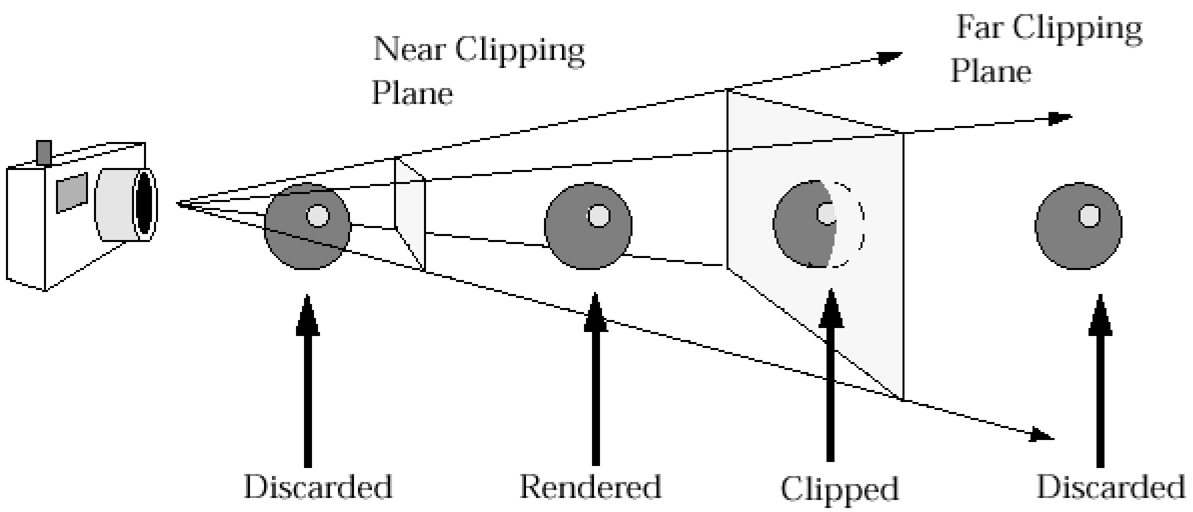
3-D Clipping

- Clipping removes objects that will not be visible
- The point of this is to remove computational effort
- Achieved in two basic steps
  - Discard objects that can't be viewed
    - i.e. objects that are behind the camera, outside the field of view, or too far away
  - Clip objects that intersect with any clipping plane



3-D Clipping

- Discarding objects that cannot possibly be seen involves comparing an objects bounding box/sphere against the dimensions of the view volume
  - Can be done before or after projection



When Do We Clip?

- We perform clipping after normalisation are done
- So, we have the following:

$$\begin{bmatrix} x_h \\ y_h \\ z_h \\ h \end{bmatrix} = M_{Norm} \cdot \begin{bmatrix} x_e \\ y_e \\ z_e \\ w_e \end{bmatrix}$$

- We apply all clipping to these homogeneous coordinates

How to Clip

- Because we have a normalised clipping volume we can test for these regions as follows:

$$-1 \le \frac{x_h}{h} \le 1 \qquad -1 \le \frac{y_h}{h} \le 1 \qquad -1 \le \frac{z_h}{h} \le 1$$

- Rearranging these we get:

$$-h \le x_h \le h \qquad -h \le y_h \le h \qquad -h \le z_h \le h \qquad if\ h > 0$$

$$h \le x_h \le -h \qquad h \le y_h \le -h \qquad h \le z_h \le -h \qquad if\ h < 0$$

- Objects that are partially within the viewing volume need to be clipped – just like the 2D case